An Axiomatization of Family Resemblance

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Abstract. We invoke concepts from the theory of hypergraphs to give a measure of the closeness of family resemblance, and to make precise the idea of a composite likeness. It is shown that for any positive integer m, for any general term possessing any extent of family resemblance strictly greater than m, there is a taxonomical representation of the term whereby each subordinate taxon has an extent of family resemblance strictly greater than m.

1 The Basic Idea

The idea of family resemblance was introduced by Wittgenstein[1] as an ingredient of his account of what constitutes possession of a concept, and what is required for the application of a general term. The account is intended to be more satisfactory than corresponding accounts that rely upon the apprehension of essential properties:

> 66. . . . Consider for example the proceedings we call games. . .if you look at them you will not see something that is common to *all*, but similarities, relationships, and a whole series of them at that. . . look for example at board games with their multifarious relationships. Now pass to card-games; here you find many correspondences to the first group, but many common features drop out, and others appear. When we pass next to ball games, much that is common is retained, but much is lost.– Are they all 'amusing'? Compare chess with noughts and crosses. Or is there always winning and losing, or competition between players? In ball games there is winning and losing, but when a child throws his ball at the wall and catches it again, this feature has disappeared. . .
>
> And the result of this examination is: we see a complicated network of similarities overlapping and criss-crossing: sometimes overall similarities, sometimes similarities of detail.
>
> 67. I can think of no better expression to characterize these similarities than "family resemblances"; for the various resemblances between members of a family: build, features, colour of eyes, gait, temperament, etc. etc. overlap and criss-cross in the same way.–And I say 'games' form a family.[1]

A *family*, as Wittgenstein envisaged the notion, can be represented as a collection of collections of properties or attributes, satisfying unspecified intersection requirements. We may assume that the properties in question are pairwise-independent, that is, that for pairs of these properties, having the one does not entail having the other. Of course there may well be properties shared by all the members of a family, for biological families, the property of being a biological item or having a common ancestor would be one such, but this property, even if it is not entailed by other properties of the collection, does not seem to be part of Wittgenstein's conception. If Rosch and Mervis have got it right[2], the problem with such properties is that not that they are shared by every member of the family, but that they are shared with members of other distinct families to the same extent. And if Wittgenstein had in mind to explicate what he took to be a common notion, then the research reported in [2] would seem to bear him out. At any rate, there are good grounds for restricting our attention to families $\mathcal{F}$ having the property that all pairs of properties in $\cup\mathcal{F}$ are independent. Wittgenstein:

> But if someone wished to say: "There is something common to all these constructions–namely the disjunction of all their common properties"– I should reply: Now you are playing with words. One might as well say: "Something runs through the whole thread–namely the continuous overlapping of those fibres". [1], para. 67.

Again, one must not try to be more precise than we have been in the general account as to what these intersection requirements are, since the collection of families is itself a family. One family might be characterized by one intersection property, another by another. In what follows, therefore, we cannot claim to do justice to the vagueness of the general notion: in the nature of the case there could never be grounds for a claim that one had.

In [2], Rosch and Mervis confirm the hypothesis that "members of categories which are considered most prototypical are those with most attributes in common with other members of the category and least attributes in common with other categories"[2] Accordingly, [2] represents the first empirical documentation of the existence in natural language categories of such general structural relationships as Wittgenstein posits. They write:

> [W]e viewed natural semantic categories as networks of overlapping attributes; the basic hypothesis was that members of a category come to be viewed as prototypical of the category as a whole in proportion to the *extent* to which they bear a family resemblance to (have attributes which overlap those of) other members of the category.[2][emphasis ours]

This notion of the *extent*, or *level* as we sometimes say, of family resemblance is what this paper is all about. Wittgenstein's account suggests, and Rosch and Mervis's confirms, that there is a logic of categories and general terms which resists conventional essentialist representation. This raises the question of whether there is an adequate non-essentialist formal representation. In this article, we propose a model of category structure that is intended to approximate what Wittgenstein, and Rosch and Mervis, have in mind; it is such that any concept, possessing any extent of family resemblance above a certain degree can be represented as a taxon, in a hierarchy of concepts, subordinating only taxa which also possess an extent of family resemblance above this degree. To show this we introduce a derivational system consisting of a base set of properties together with a collection of rules for generating taxa. Along the way we show how to define a measure of closeness of family resemblance, and we illustrate the relationship between family resemblance and the mathematical theory of hypergraphs by making precise the notion of a composite likeness.

2 Its Realization

A collection of objects, X forms a family, $\mathcal{F}$, in virtue of some set, $\mathcal{P}$, of properties, such that $\forall x \in X, \exists \mathcal{P}' \subseteq \mathcal{P} : \forall \varphi \in \mathcal{P}', \varphi x$. But typically, the application of the term *family* requires that $\mathcal{P}$ be sufficiently small in relation to X, that q-tuples of objects $(0 < q \leq |X|)$ of X share properties from $\mathcal{P}$. Hence the informal notion of *family resemblance*, the physically apparent *harmoniousness* of families, drawn, as it were, from a restricted palette of features. Informally, on this account, a family is represented as a collection of collections of properties that, to some extent, overlap. The harmoniousness of a family lies in the character of this overlap.

Definition 1 *Let $\mathcal{P}$ be a set of properties. Then a set $\mathcal{F}$ is a* family *on $\mathcal{P}$ if $\mathcal{F}_{\neq\emptyset} \subseteq 2^{\mathcal{P}}$ and $\emptyset \notin \mathcal{F}$.*[1]

A word is in order about the consistency of this set-theoretic conception of a family with Wittgenstein's view of the indeterminacy of concepts. There are at least two aspects of this indeterminacy; one is that some concepts are "unbounded"; a second, related, issue involves cases which are not clearly instances of a general term[1]. There is a difference between a collection with indeterminately many members, and a collection in which membership is indeterminate. A set, however, is typically construed as being a collection for which both membership and cardinality are fixed.

Part of this apparent incongruity can be resolved by allowing the set $\mathcal{P}$ of properties to be indefinitely large. In this way the size of a family may be indefinite, and the issue of the boundaries of a concept needn't for practical purposes arise. As for the second kind of indeterminacy, apparently pertaining to vagueness, the model we present is intended as a discrete approximation of a potentially continuous phenomena, as, for example, a binomial distribution can be used to approximate a normal distribution, and therefore the model shouldn't be expected to exactly replicate the natural continuity of general terms.

Now as 'family resemblance' refers to a pattern of intersections among the members of a family, there are two dimensions along which the general notion

[1] Reference to $\mathcal{P}$ is omitted when context is sufficiently disambiguating.

may be analyzed, and which our account must make salient if it is to be adequate with the respect to the notion envisaged by Wittgenstein. In addition to the frequency with which an overlap of attributes between items occurs, one can speak of the *thickness* of the overlap, or the number of elements of which the overlap is comprised. This latter quality can be expressed as a generalization of the former, which we measure using the *harmonic number* of a family.

Definition 2 *If S is a set and q is a positive integer, we write '$\binom{S}{q}$' for the set of all q-tuple subsets of S. If $\mathcal{F}$ is a family then the* harmonic number *of $\mathcal{F}$, $\eta(\mathcal{F})$, is defined:*

$$\eta(\mathcal{F}) := \begin{cases} \mathit{min}\ n \in \mathbb{Z}^+ : \exists \mathcal{G} \in \binom{\mathcal{F}}{n} : \cap \mathcal{G} = \emptyset & \text{if this limit exists;} \\ \infty & \text{otherwise.} \end{cases} \tag{2.1}$$

*If $\eta(\mathcal{F}) > n$ then we say that $\mathcal{F}$ is n-*harmonic.

For example, let

$$\mathcal{F} = \{\{1,2\},\{1,3\},\{2,3\}\}$$

Then $\eta(\mathcal{F}) = 3$. This is because every pair of elements of $\mathcal{F}$ has a non-empty intersection while there is a triple of edges, namely, $\mathcal{F}$ itself, whose intersection is empty. Another family whose harmonic number is 3 is the following:

$$\{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\}$$

Note that if a finite family $\mathcal{F}$ is such that $\cap\mathcal{F} = \emptyset$, then for some positive integer m where $2 \leq m \leq |\mathcal{F}|$, $\eta(\mathcal{F}) = m$, and if $\cap F \neq \emptyset$ then $\eta(\mathcal{F}) = \infty$.

Definition 3 *A family $\mathcal{F}$ is* trivial *if $\cap\mathcal{F} \neq \emptyset$.*

For example, $\{\{1\}\}$ is trivial, as is the family $\{e \subseteq \mathbb{Z}^+ \mid 8 \in e\}$.

Proposition 1 $\forall \mathcal{F}, [\mathcal{F}$ *is not trivial* $\Leftrightarrow \forall x \in \cup\mathcal{F}, \exists e \in \mathcal{F} : e \subseteq \cup\mathcal{F} - \{x\}]$.

Generalizing n-harmonics we have:

Definition 4 *If $\mathcal{F}$ is a family and n is a positive integer, the n(-*harmonic*)* saturation number *of $\mathcal{F}$, $\sigma_n(\mathcal{F})$, is defined:*

$$\sigma_n(\mathcal{F}) := \mathit{min}\ m \geq 1 : \exists k \in [n], \exists \mathcal{G} \in \binom{\mathcal{F}}{k} : |\cap \mathcal{G}| < m$$

where for any positive integer n, '$[n]$' abbreviates '$\{1,2,...,n\}$'. If $\sigma_n(\mathcal{F}) > m$ then $\mathcal{F}$ is m n-(harmonically) saturated.

The idea behind the n-saturation number of a family $\mathcal{F}$ is this: Informally, let the *thickness* (*thinness*) of a k-tuple be the size of its intersection. Then the n-saturation number of $\mathcal{F}$ is 1 larger than the thickness of the thinnest k-tuple of $\mathcal{F}$ for all $k \in [n]$. It can be seen to follow from this informal reading of '$\sigma_n(\mathcal{F})$' that if $\sigma_n(\mathcal{F}) > m \geq 1$ then for each $k \in [n]$ $(n \geq 1)$, every k-tuple subset of $\mathcal{F}$ is at least m thick. The converse is also true:

Proposition 2 $\forall \mathcal{F}, n \geq 1, m \geq 1,\ [\sigma_n(\mathcal{F}) > m \Leftrightarrow \forall k \in [n], \forall \mathcal{G} \in \binom{\mathcal{F}}{k}, |\cap \mathcal{G}| \geq m]$.

For example, consider the following families:

$$\mathcal{F}_1 = \{\{1,2\},\{1,3\},\{2,3\}\}$$
$$\mathcal{F}_2 = \{\{1,2,4\},\{2,3,5\},\{3,4,1\},\{4,5,2\},\{5,1,3\}\}$$
$$\mathcal{F}_3 = \{\{1,2,3\}\}$$
$$\mathcal{F}_4 = \{\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\},\{2,3,4,5\}\}$$

If $n = 1$ then $\sigma_n(\mathcal{F}_1) = 3, \sigma_n(\mathcal{F}_2) = 4 = \sigma_n(\mathcal{F}_3)$, and $\sigma_n(\mathcal{F}_4) = 5$. In addition we have $\sigma_2(\mathcal{F}_1) = 2, \sigma_2(\mathcal{F}_2) = 2, \sigma_2(\mathcal{F}_3) = 4, \sigma_2(\mathcal{F}_4) = 4, \sigma_3(\mathcal{F}_2) = 1, \sigma_3(\mathcal{F}_4) = 3, \sigma_4(\mathcal{F}_1) = 1 = \sigma_5(\mathcal{F}_4)$, and $\sigma_4(\mathcal{F}_4) = 2$.

Immediate from the definitions of harmonic number and harmonic saturation, the next proposition illustrates the sense in which m n-saturation is a generalization of n-harmonicity:

Proposition 3 $\forall \mathcal{F}, n \geq 1,\ [\eta(\mathcal{F}) > n \Leftrightarrow \sigma_n(\mathcal{F}) > 1]$.

The following theorem asserts that if $\mathcal{F}$ is a non-trivial n-harmonic family, $\mathcal{F}$ is m n-saturated only if, for all $k \in [n]$, the thickness of k-tuples of $\mathcal{F}$ increases as k decreases.

Theorem 4 $\forall \mathcal{F}, n \geq 1, m \geq 1$, *[if $\mathcal{F}$ is not trivial then* $[\sigma_n(\mathcal{F}) > m \Rightarrow \forall i, \forall \mathcal{G} \in \binom{\mathcal{F}}{n-i}, |\cap \mathcal{G}| \geq i + 1\ (0 \leq i \leq n-1)]]$.

Proof. Assume that $\mathcal{F}$ is not trivial and that $\sigma_n(\mathcal{F}) > m \geq 1$. Let i be arbitrary $(0 \leq i \leq n-1)$. Suppose that $\mathcal{G} \in \binom{\mathcal{F}}{n-i}$ is such that $|\cap \mathcal{G}| < i + 1$. Let $\cap \mathcal{G} = \{x_1, x_2, ..., x_h\} (h \leq i)$. Let $\mathcal{J} = \{\cup \mathcal{F} - \{x_m\} \mid m \in [h]\}$. From Proposition 1, since $\mathcal{F}$ is not trivial, $\forall e \in \mathcal{J}, \exists f \in \mathcal{F} : f \subseteq e$. But $\cap(\mathcal{G} \cup \mathcal{J}) = \emptyset$. $\therefore\ \exists g \in [(n-i)+h] \cap [n], \exists \mathcal{H} \in \binom{\mathcal{F}}{g}$ such that $\mathcal{G} \subseteq \mathcal{H}$ and $\cap \mathcal{H} = \emptyset$. $\therefore\ \eta(F) \leq n$, which is absurd, given Proposition 3.

The notions of harmonic number and harmonic saturation represent two dimensions in terms of which extent of family resemblance can be analysed. One, corresponding to harmonic number, refers to the frequency with which attributes are shared among the members of a family; the second, corresponding to harmonic saturation, pertains to the extent to which attributes are shared, relative to a given frequency. As a result of its dyadic character, there are some families which are apparently not comparable with respect to family resemblance. Consider the case, for example, where a family $\mathcal{F}_1$ has a lower harmonic number than a family $\mathcal{F}_2$ while for some $k \geq 1$, every k-tuple subset of $\mathcal{F}_1$ is thicker than all, or even most, k-tuple subsets of $\mathcal{F}_2$. For example, let

$$\mathcal{F}_1 = \{\{a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2,c_3\},\{d_1,d_2,d_3,a_1,a_2,a_3,b_1,b_2,b_3\},$$
$$\{d_1,d_2,d_3,a_1,a_2,a_3,c_1,c_2,c_3\},\{d_1,d_2,d_3,b_1,b_2,b_3,c_1,c_2,c_3\}\}$$
$$\mathcal{F}_2 = \{\{1,2,3,4\},\{1,2,3,5\},\{1,3,4,5\},\{2,3,4,5\},\{1,2,4,5\}\}.$$

Then $\eta(\mathcal{F}_1) = 4$ and $\sigma_3(\mathcal{F}_1) > 3$, and $\eta(\mathcal{F}_2) = 5$, while $\sigma_3(\mathcal{F}_2) = 3$. Because of the inherent vagueness of 'family resemblance' it would seem that prima facie we have no grounds for saying of either family that it possesses a greater (lesser) degree of family resemblance than the other, nor that their respective levels of family resemblance are equal. Such considerations suggest that if we are to measure family resemblance, then our gauge should be relativized to a given frequency. Accordingly we propose:

Definition 5 *For a family $\mathcal{F}$ and positive integer n, the* n-resemblance *of $\mathcal{F}$ is $\sigma_n(\mathcal{F})$.*

Proposition 5 $\forall \mathcal{F}, n \geq 1, 1 \leq \sigma_n(\mathcal{F}) \leq |\cup \mathcal{F}| + 1$. *If $\mathcal{F}$ is not trivial then $\sigma_n(\mathcal{F}) \leq |\cup \mathcal{F}|$.*

Proof. Suppose that $\sigma_n(\mathcal{F}) > |\cup \mathcal{F}| + 1$. Then $\forall k \in [n]$, every k-tuple of $\mathcal{F}$ is at least $|\cup \mathcal{F}| + 1$ thick, which is absurd. Now suppose that $\mathcal{F}$ is not trivial, and let $\sigma_n(\mathcal{F}) > |\cup \mathcal{F}|$. Then $\forall k \in [n]$, every k-tuple subset of $\mathcal{F}$ is at least $|\cup \mathcal{F}|$ thick. But this can be so only if $|\mathcal{F}| = 1$, in which case $\mathcal{F}$ is trivial, contrary to supposition.

Using harmonic saturation we can define a relation of *closeness* of family resemblance:

Definition 6 *A family $\mathcal{F}$* more closely n-resembles $\mathcal{G}_1$ than $\mathcal{G}_2$ *if $\sigma_n(\mathcal{F} \cup \mathcal{G}_1) > \sigma_n(\mathcal{F} \cup \mathcal{G}_2)$.*

Thus, for example, where

$$\begin{aligned} \mathcal{F} =& \{\{5,3,4\},\{6,3,4\}\} \\ \mathcal{G}_1 =& \{\{6,5,1,2,3\},\{6,5,1,2,4\},\{1,3,4\},\{2,3,4\}\} \\ \mathcal{G}_2 =& \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\} \end{aligned}$$

$\mathcal{F}$ more closely 3-resembles $\mathcal{G}_1$ than $\mathcal{G}_2$ because $\sigma_3(\mathcal{F} \cup \mathcal{G}_1) = 2$ and $\sigma_3(\mathcal{F} \cup \mathcal{G}_2) = 1$. Also, $\mathcal{F}$ more closely 2-resembles $\mathcal{G}_1$ than $\mathcal{G}_2$ since $\sigma_2(\mathcal{F} \cup \mathcal{G}_1) = 3$ and $\sigma_2(\mathcal{F} \cup \mathcal{G}_2) = 2$. However, $\mathcal{F}$ does not more closely 1-resemble $\mathcal{G}_1$ than $\mathcal{G}_2$ since $\sigma_1(\mathcal{F} \cup \mathcal{G}_1) = \sigma_1(\mathcal{F} \cup \mathcal{G}_2) = 4$.

Alternatively, we can define a measure for the similarity of families with respect to family resemblance:

Definition 7 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be families. Then $\mathcal{F}_1$* d-n-resembles *$\mathcal{F}_2 \Leftrightarrow \sigma_n(\mathcal{F}_1 \cup \mathcal{F}_2) \geq \sigma_n(\mathcal{F}_1) - d$ $(0 \leq d \leq \sigma_n(\mathcal{F}_1))$.*

For instance, let

$$\begin{aligned} \mathcal{F}_1 =& \{\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\},\{2,3,4,5\}\} \\ \mathcal{F}_2 =& \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}. \end{aligned}$$

Then $\sigma_3(\mathcal{F}_1) = 3, \sigma_3(\mathcal{F}_2) = 2$, and $\sigma_3(\mathcal{F}_1 \cup \mathcal{F}_2) = 2$. Therefore the least value of d such that $\mathcal{F}_1$ d-3-resembles $\mathcal{F}_2$ is 1. In general, the higher the least value of d is, the greater the degree with which $\mathcal{F}_2$ attenuates the n-resemblance of $\mathcal{F}_1$ when the two families are juxtaposed. To take another example, let

$$\mathcal{F}_1 = \{\{1,2,3,4\}\}$$
$$\mathcal{F}_2 = \{\{4,5,6\},\{1,4,5\},\{1,6\}\}$$

Then 4 is the least value of d such that $\mathcal{F}_1$ d-3-resembles $\mathcal{F}_2$ because $\sigma_3(\mathcal{F}_1 \cup \mathcal{F}_2) = 1$ and $\sigma_3(\mathcal{F}_1) = 5$, while $\mathcal{F}_2$ 0-3-resembles $\mathcal{F}_1$ because $\sigma_3(\mathcal{F}_2) = 1$.

Proposition 6 $\forall \mathcal{F}_1, \mathcal{F}_2, n \geq 1, \mathcal{F}_1$ *0-n-resembles* $\mathcal{F}_2 \Leftrightarrow \sigma_n(\mathcal{F}_1 \cup \mathcal{F}_2) = \sigma_n(\mathcal{F}_1)$.

3 Composite Families

In literature, the term *composite* is applied to fictional characters who comprise traits of numerous source figures. In our account of family resemblance we apply the term to what could be regarded as the formal counterpart of such a fictional character. The idea of a composite, mathematically realized, is the lynch pin connecting the study of families as conceived by Wittgenstein and empirically studied by Rosch and Mervis to the well-established mathematical theory of hypergraphs. In fact, save that for local purposes we understand $\mathcal{P}$ as a set of properties, the language of *families* in our account could yield its place to the language of *hypergraphs*, and the *harmonics of families* could be understood abstractly as defining hitherto unstudied properties of hypergraphs. In fact, the fundamental features that ground the idea of a family are dual to the characteristics that define one of the principal mathematical interests in hypergraphs: the properties relating to *chromaticity.*

Definition 8 *A set* c *is a* composite *of a family* $\mathcal{F}$ *iff* $\forall e \in \mathcal{F}, \mathrm{c} \cap e \neq \emptyset$; c *is a minimal composite of* $\mathcal{F}$ *iff* c *is a composite of* $\mathcal{F}$ *and no proper subset of* c *is a composite of* $\mathcal{F}$*. The* composite family $\mathcal{C}(\mathcal{F})$ *of* $\mathcal{F}$*, which may be written '*$\mathcal{CF}$*', is the set of all minimal composites of* $\mathcal{F}$*.*

Proposition 7 $\forall \mathcal{F}, [\mathcal{CC}(\mathcal{F}) \subseteq \mathcal{F}]$*, and* $[\forall e \in \mathcal{F}, \exists f \in \mathcal{CC}(\mathcal{F}) : e \supseteq f]$.

Definition 9 *If* $\mathcal{F}$ *is a family and* m *is a positive integer, a function* $f : \cup\mathcal{F} \to [m]$ *is an* m-colouring *of* $\mathcal{F}$ *if* $\forall e \in \mathcal{F}, k \in [m], e \not\subseteq \{x \in \cup\mathcal{F} \mid f(x) = k\}$*. If there is an* m*-colouring of* $\mathcal{F}$ *then we say that* $\mathcal{F}$ *is* m-colourable*;* $\mathcal{F}$ *is* m-uncolourable*, else. The* chromatic number *of* $\mathcal{F}$*,* $\chi(\mathcal{F})$*, is defined:*

$$\chi(\mathcal{F}) := \begin{cases} \textit{min } m \in \mathbb{Z}^+ : \mathcal{F} \textit{ is } m\textit{-colourable if this limit exists;} \\ \infty \qquad \textit{otherwise.} \end{cases} \tag{3.1}$$

The chromatic number of a family can also be defined in terms of the set of decompositions of its union.

Definition 10 *Let S be a set and m a positive integer. The* set of m-decompositions *of S, $\Delta_m(S)$, is defined:*

$$\Delta_m(S) := \{\delta = \{d_1, ..., d_m\} \mid \bigcup_{i=1}^{m} d_i = S\}$$

Proposition 8 $\forall \mathcal{F}, \forall m \geq 1, [\chi(\mathcal{F}) > m \Leftrightarrow \forall \delta \in \Delta_m(\cup\mathcal{F}), \exists d \in \delta, e \in \mathcal{F} : e \subseteq d]$.

Definition 11 *Let $\mathcal{F}$ be a family and S a subset of $\cup\mathcal{F}$. The* restriction of $\mathcal{F}$ to S, *$\mathcal{F}[S]$, is the family defined:*

$$\mathcal{F}[S] := \{e \in \mathcal{F} \mid e \subseteq S\}$$

Theorem 9 $\forall \mathcal{F}, n \geq 1, m \geq 1, [\sigma_n(\mathcal{F}) > m \Leftrightarrow . \forall S, |S| < m \Rightarrow \chi((\mathcal{C}(\mathcal{F})[\cup\mathcal{F} - S])) > n]$.

Proof. Let $\mathcal{F}$ be an arbitrary family, and let m and n be arbitrary positive integers.
[$\Rightarrow$] Let $S \subseteq \cup\mathcal{F}$ be such that $|S| < m$ and $\mathcal{C}(\mathcal{F})[\cup\mathcal{F} - S]$ is n-colourable. Then $\exists \delta \in \Delta_n(\cup\mathcal{F} - S)$ such that $\forall e \in \mathcal{C}(\mathcal{F})$, $\forall d \in \delta$, $e \not\subseteq d$ (Proposition 8). Let $\mathcal{G} = \{\cup\mathcal{F} - d \mid d \in \delta\}$. Then $\forall g \in \mathcal{G}, g$ is a composite of $\mathcal{C}(\mathcal{F})$, and $\cap\mathcal{G} = S$. Therefore $\exists k \in [n], \exists \mathcal{J} \in \binom{\mathcal{CC}(\mathcal{F})}{k} : \cap\mathcal{J} \subseteq S$. But $\mathcal{CC}(\mathcal{F}) \subseteq \mathcal{F}$ (Proposition 7). Therefore $\sigma_n(\mathcal{F}) \leq m$.
[$\Leftarrow$] Let $\sigma_n(\mathcal{F}) = k \leq m$. Then $\exists j \in [n]$, $\exists \mathcal{G} \in \binom{\mathcal{F}}{j} : |\cap\mathcal{G}| < k \leq m$. Let $\delta = \{\cup\mathcal{F} - g \mid g \in \mathcal{G}\}$. Then $\delta \in \Delta_j(\cup\mathcal{F} - \cap\mathcal{G})$. But $\forall e \in \mathcal{C}(\mathcal{F})[\cup\mathcal{F} - \cap\mathcal{G}], \forall d \in \delta, e \not\subseteq d$ (Proposition 7). Therefore $\mathcal{C}(\mathcal{F})[\cup\mathcal{F} - \cap\mathcal{G}]$ is n-colourable (Proposition 8). But $|\cap\mathcal{G}| < m$. Therefore $\exists S \subseteq \cup\mathcal{F} : |S| < m$ and $\chi(\mathcal{C}(\mathcal{F})[\cup\mathcal{F} - S]) \leq n$.

For the case $m = 1$, what Theorem 9 amounts to in the presence of Proposition 3 is the assertion that n-harmonicity in a family is equivalent to the n-uncolourability of its composite family. If $m \geq 1$ then Theorem 9 asserts that the n-uncolourability of a composite family is preserved under the deletion of fewer than m elements from $\cup\mathcal{F}$ if $\mathcal{F}$ is m n-harmonically saturated.

4 Taxonomic Hierarchies

Rosch and Mervis documented the cognitive significance of distinct families within more generic groupings of items[2]. As an example, the family of 'pets' lies within, or is subordinated by, the more general category of 'domestic animals', a category which also comprises families of otherwise subordinated non-human creatures. Similarly, the concepts 'dog', 'cat', 'rabbit' are members of the superordinating category 'pets'.

For Rosch and Mervis, as for Wittgenstein, what accounts for the subordination of a concept within a more general category is not that there is some single criterion possessed by all and only members of the category, but rather that there is a network of shared attributes, or intersections. This network is the family resemblance of the category, whose extent we have represented using the concept of the n-resemblance of a family $\mathcal{F}$, which refers to $\sigma_n(\mathcal{F})$, the n-saturation number of $\mathcal{F}$.

Now granted that subordinate categories inherit the extent of family resemblance of superordinating ones, is it true that for every category, with any level of family resemblance, there is a taxonomic representation which preserves this fact? Below, we show one way that a subordination relation can be structured so that this is answered affirmatively.

Definition 12 *A taxon is a taxonomic group of any rank, including all the subordinate groups; it is any group of organisms or populations considered to be sufficiently distinct from other such groups to be treated as a separate unit. The (taxonomic) rank of a taxon is its position in a hierarchy of classification.[3]*

To formalize what is meant by the *rank* of a taxon, we employ the notion of an *m-n-derivation*. Intuitively, this can be understood analogously to a proof in a logical system, substituting hypergraph and set-theoretic operations for rules of inference. Essentially we take iterations of these operations to structure the subordination relation among taxa. The system as a whole will be shown to be sound and complete with respect to n-resemblance strictly greater than $m \geq 1$.

Definition 13 *Let $\mathcal{F}$ and $\mathcal{G}$ be families. If every element of $\mathcal{F}$ is a superset of an element of $\mathcal{G}$, then $\mathcal{F}$* subsumes *$\mathcal{G}$, written '$\mathcal{F} \sqsupseteq \mathcal{G}$', or '$\mathcal{G} \sqsubseteq \mathcal{F}$'.*

Proposition 10 $\forall m \geq 1, n \geq 1, \forall \mathcal{F}, \mathcal{G}$, *[if $\sigma_n(\mathcal{F}) > m$ and $\mathcal{G} \sqsupseteq \mathcal{F}$ then $\sigma_n(\mathcal{G}) > m$].*

Given our intention to devise a system which is sound with respect to n-resemblance, Proposition 10 licenses the following rule 'upward subsumption':

$$[\uparrow\sqsupseteq]: \text{ given } \mathcal{F}, \text{ if } \mathcal{G} \sqsupseteq \mathcal{F}, \text{ obtain } \mathcal{G} \tag{4.1}$$

Definition 14 *Let $S = \mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_i, ..., \mathcal{F}_q$ be a sequence of q families $(1 < q)$, and let T be a set. Then if $n \geq 1$, T* n-covers *S if $\exists\{\mathcal{F}_1, ..., \mathcal{F}_n\} \in \binom{S}{n}$ such that $\forall i \in [n]$, $\exists e \in \mathcal{F}_i : T \supseteq e$; T* minimally n-covers *S if T n-covers S, and no proper subset of T n-covers S. We write '$\frac{n}{q}(\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_q)$' for the set of all minimal n-covers of S.*

Theorem 11 $\forall m \geq 1, n \geq 1$, *for any sequence $\mathcal{G}_1, ..., \mathcal{G}_{n+1}$ of families, if for each $i \in [n+1]$, $\sigma_n(\mathcal{G}_i) > m$, then $\sigma_n(\frac{n}{n+1}(\mathcal{G}_1, ..., \mathcal{G}_{n+1})) > m$.*

Proof. Let $\sigma_n(\frac{n}{n+1}(\mathcal{G}_1, ..., \mathcal{G}_{n+1})) = k \leq m$. Then $\exists j \in [n], \mathcal{F} \in \binom{\frac{n}{n+1}(\mathcal{G}_1, ..., \mathcal{G}_{n+1})}{j}$: $|\cap \mathcal{F}| < m$. By definition, each $e \in \mathcal{F}$ is a superset of an element from every member of some n-tuple subset of $\mathcal{G}_1, ..., \mathcal{G}_{n+1}$. By a pigeonhole argument, $\exists i \in [n+1]$ such that $\forall e \in \mathcal{F}$, $\exists f \in \mathcal{G}_i$: $e \supseteq f$. So if $\sigma_n(\mathcal{G}_i) > m$ then $|\cap \mathcal{F}| \geq m$: a contradiction. Whence $\exists i \in [n+1], \sigma_n(\mathcal{G}_i) \leq m$.

Because $\frac{n}{n+1}$ preserves n-resemblance strictly greater than $m \geq 1$ (Theorem 11), in addition to rule [$\uparrow\sqsupseteq$], we also therefore have 'n over $n+1$':

$$[\frac{n}{n+1}]: \text{ Given } \mathcal{G}_1, ..., \mathcal{G}_{n+1}, \text{ obtain } \frac{n}{n+1}(\mathcal{G}_1, ..., \mathcal{G}_{n+1}) \ (n \geq 1). \tag{4.2}$$

Our final rule is intended to license type-raising for m-tuples from a base set $\mathcal{P}$ of properties:

$$[m]: \text{ From } \{x_1, x_2, ..., x_m\} \subseteq \mathcal{P} \text{ obtain } \{\{x_1, x_2, ..., x_m\}\} \ (m \geq 1). \tag{4.3}$$

Definition 15 *If $n \geq 1$ and $m \geq 1$, an m-n-*derivation *of a family $\mathcal{F}$ from a set $\mathcal{P}$ is a finite sequence of families on $\mathcal{P}$, ending with $\mathcal{F}$, where each family is obtained either from preceding ones by an application of $[\frac{n}{n+1}]$ or [$\uparrow\sqsupseteq$], or from $\mathcal{P}$ by an application of $[m]$.*

Theorem 12 $\forall \mathcal{F}, n \geq 1, m \geq 1$, *there is an m-n-derivation of $\mathcal{F} \Rightarrow \sigma_n(\mathcal{F}) > m$.*

Proof. It is sufficient to prove that $[\frac{n}{n+1}]$ and [$\uparrow\sqsupseteq$] preserve n-resemblance strictly greater than m, and that $\forall x_1, x_2, ..., x_m \in \mathcal{P}, \sigma_n(\{\{x_1, x_2, ..., x_m\}\}) > m$. We have already shown the former (Proposition 10 and Theorem 11); the latter follows from the fact that if $\{x_1, ..., x_m\} \subseteq \mathcal{P}$, then $\sigma_n(\{\{x_1, ..., x_m\}\}) = m + 1$.

Theorem 13 $\forall \mathcal{F}, n \geq 1, m \geq 1, \sigma_n(\mathcal{F}) > m \Rightarrow$ *there is an m-n-derivation of $\mathcal{F}$.*

Proof. Let $m \geq 1$ and $n \geq 1$ be arbitrary. Let $\mathcal{F}$ be an arbitrary family on a set $\mathcal{P}$ of properties such that $\sigma_n(\mathcal{F}) > m$. We induce on $|\mathcal{F}|$. For the basis, let $|\mathcal{F}| \leq n$. Then $|\cap \mathcal{F}| \geq m$. Let $\{x_1, ..., x_m\} \subseteq \cap\mathcal{F}$. Then $\mathcal{F} \sqsupseteq \{\{x_1, ..., x_m\}\}$. Therefore there is an m-n-derivation of $\mathcal{F}$ from $\mathcal{P}$ using $[m]$ and an application of [$\uparrow\sqsupseteq$].

Now let $|\mathcal{F}| \geq n + 1$. Where $\{e_1, ..., e_{n+1}\} \subseteq \mathcal{F}$, define:

$$\mathcal{F}_i := \mathcal{F} - \{e_i\} \qquad (i \in [n+1])$$

Then $\forall i \in [n+1], \sigma_n(\mathcal{F}_i) > m$, by the downward monotonicity of n-resemblance strictly greater than m. The hypothesis of induction therefore allows us to assert that for each $i \in [n+1]$, there is an m-n-derivation of $\mathcal{F}_i$ from $\mathcal{P}$. But $\forall e \in \mathcal{F}$, e is an n-cover for $\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_{n+1}$. Therefore $\mathcal{F} \sqsupseteq \frac{n}{n+1}(\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_{n+1})$. Whence, there is an m-n-derivation of $\mathcal{F}$ from $\mathcal{P}$, namely, the sequence consisting of the m-n-derivation of $\mathcal{F}_1$, followed by the m-n-derivation of $\mathcal{F}_2$, ..., followed by the m-n-derivation of $\mathcal{F}_{n+1}$, followed by an application of $\frac{n}{n+1}$, and terminated by an application of [$\uparrow\sqsupseteq$].

With the notion of an m-n-derivation in hand, we may now speak of the *rank* of a taxon.

Definition 16 *The* rank *of a family $\mathcal{F}$, relative to a given m-n-derivation $\mathfrak{D}$ of a family $\mathcal{G}$, $\rho_{\mathfrak{D}}(\mathcal{F})$, is the position of $\mathcal{F}$ in $\mathfrak{D}$.*

We also introduce the notion of a *proper* (m-n-)derivation to distinguish between useful and irrelevant applications of rules. We shall not have occasion here to provide a comprehensive analysis of relevance with respect to derivations, and rely on what is, we hope, a shared prima facie intuition with the reader.

Definition 17 *An m-n-derivation $\mathfrak{D} = (\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_i, ..., \mathcal{G}_q)$ of a family $\mathcal{G}_q$ ($q \geq 1$) is* proper *if $\forall i \in [q], \mathfrak{D} - (\mathcal{G}_i)$ is not an m-n-derivation of $\mathcal{G}_q$.*

Evidentally, Theorem 13 may be restated in terms of proper derivations. Definition 17 enables us to formalize a concept of taxonomical subordination.

Definition 18 *A family $\mathcal{F}$ m-n-*subordinates *a family $\mathcal{G}$ iff there is a proper m-n-derivation of $\mathcal{F}$ in which $\mathcal{G}$ appears.*

Theorem 14 $\forall \mathcal{F}, m \geq 1, n \geq 1, \sigma_n(\mathcal{F}) > m \Rightarrow \exists q \geq 1$*: there is a representation of $\mathcal{F}$ as a taxon of rank q where $\mathcal{F}$ m-n-subordinates only taxa of n-resemblance strictly greater than m.*

Proof. Theorem 14 is immediate from the fact that $\forall \mathcal{F}$, if $\sigma_n(\mathcal{F}) > m$ then there is a proper m-n-derivation $\mathfrak{D}$ of $\mathcal{F}$ such that $\forall \mathcal{G}$, if $\mathcal{G}$ is in $\mathfrak{D}$ then $\sigma_n(\mathcal{G}) > m$ (Theorems 12 and 13).

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